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AUTHOR(S):

MAESONO, Hisatomo

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## Some remark on weak dividing

前園 久智 (Hisatomo MAESONO)  
早稲田大学メディアネットワークセンター  
(Media Network Center, Waseda University)

### Abstract

Weak dividing has been characterized variously in simple theory.  
We try to argue about the restricted notions of it.

### 1. Weak dividing

We recall some definitions.

**Definition 1** Let  $\varphi(x_0, x_1, \dots, x_{n-1})$  be a formula and  $p(x)$  be a type. We denote the type  $\{\varphi(x_0, x_1, \dots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \dots \cup p(x_{n-1})$  by  $[p]^\varphi$ .

Let  $A \subset B$  and  $p(x) \in S(B)$ .

$p(x)$  *divides over*  $A$  if there are a formula  $\varphi(x, b) \in p(x)$  and an infinite sequence  $\{b_i : i < \omega\}$  with  $b \equiv b_i(A)$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is  $k$ -inconsistent for some  $k < \omega$ .

$p(x)$  *weakly divides over*  $A$  if there is a formula  $\varphi(\bar{x}) \in L_n(A)$  such that  $[p[A]]^\varphi$  is consistent, while  $[p]^\varphi$  is inconsistent.

In this note, we call such formula " $\varphi(\bar{x})$ " in the definition the *witness formula* of weak dividing for the sake of convenience.

I introduce examples from [2], [3].

**Example 2** Let  $T$  be the theory of dense linear order and  $p(x) = "a < x < b"$ . Then  $p(x)$  does not weakly divide over  $\emptyset$ . And let  $q(x, y) = "x < c < y"$ . Then  $q(x, y)$  weakly divides over  $\emptyset$  by the formula  $\varphi(x_1, y_1; x_2, y_2) = "y_1 < x_2"$ .

**Example 3** Let  $T$  be the theory of an equivalence relation with two infinite classes of the language  $L = \{a \text{ binary relation } E(x, y)\}$ . And let  $\models \neg E(a, b)$ . Then the type  $\text{tp}(a/b)$  does not divide over  $\emptyset$ , while  $\text{tp}(a/b)$  weakly divides over  $\emptyset$  by the formula  $\neg E(x, y)$ .

**Example 4** Let  $(V, <, >)$  be a vector space  $V$  over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let  $a, b, c$  be independent vectors in  $V$  such that  $a \perp b$ , while  $b \not\perp c$  and  $a \not\perp c$ . Then  $\text{tp}(a/bc)$  does not weakly divide over  $\emptyset$ . But  $\text{tp}(a/bc)$  weakly divides over  $c$  by the formula  $\varphi(x, y) = "x \text{ is a linear combination of } y \text{ and } c"$ .

In various characterizations, one of the most important results is the next theorem.

**Theorem 5** (Kim [3])

*$T$  is stable if and only if weak dividing is symmetric in  $T$ .*

## 2. Restricted notions of weak dividing

In examples above, we can see that witness formulas have different properties in the sense of stability theory. So I considered that we can divide witness formulas into some classes according to the properties.

**Definition 6** Let  $A \subset B$  and  $p(x) \in S(B)$ .

$p(x)$   *$\mathcal{M}$ -weakly divides over  $A$*  if there are a formula  $\varphi(\bar{x}) \in L_n(A)$  and a Morley sequence  $I = \{a_i : i < n+1\}$  of  $p|_A$  such that  $\models \varphi(a_0, a_1, \dots, a_{n-1})$ , while the type  $[p]^\varphi$  is inconsistent.

$p(x)$   *$M$ -weakly divides over  $A$*  if there are a formula  $\varphi(\bar{x}) \in L_n(A)$  and a Morley sequence  $I = \{a_i : i < n+1\}$  of  $p|_A$  such that  $\models \varphi(a_0, a_1, \dots, a_{n-1})$ , while there is no Morley sequence  $J = \{b_i : i < n+1\}$  of  $p$  over  $A$  such that  $\models \varphi(b_0, b_1, \dots, b_{n-1})$ .

If we set the sequence  $I$  indiscernible over  $A$  in the definition above, we can define  $\mathcal{I}$ -weak dividing and  $I$ -weak dividing in the same way.

I proved the next result.

**Theorem 7** *Let  $T$  be simple.*

*Then  $T$  is stable if and only if  $\mathcal{M}$ -weak dividing is symmetric in  $T$ .*

Example 3 is easy. But some of examples of  $\mathcal{M}$ -weak dividing may be variants of it.

**Fact 8** *Let  $T$  be simple. And let  $A \subset B$  and  $p(x) \in S(B)$ .*

*$p(x)$  does not divide over  $A$ , while  $p(x)$   $\mathcal{M}$ -weakly divides over  $A$  by a formula  $\varphi(x, y) \in L_2(A)$ . Then for any realization  $ab$  of  $\varphi$  with  $a \perp_A b$ ,  $L\text{stp}(a/A) \neq L\text{stp}(b/A)$ .*

I showed Fact 8 for 2-variable witness formulas. But the same Fact holds for  $n$ -variable witness formulas under the assumption that  $T$  has  $n$ -amalgamation property. ( see [4],[5] ) Once I told about some weak dividing for  $n$ -dividing in  $n$ -simple theory.

In recent years another variant of dividing, "thorn"-dividing has been characterized in rosy theory. ( see e.g. [6] ) I tried to define weak notion of  $p$ -dividing (thorn-dividing). We recall some definitions first.

**Definition 9** Let  $A \subset B$  and  $p(x) \in S(B)$ .

$p(x)$  *strongly divides over*  $A$  if there is a formula  $\varphi(x, b) \in p(x)$  such that  $b \notin \text{acl}(A)$  and  $\{\varphi(x, b_i) : b_i \models \text{tp}(b/A)\}$  is  $k$ -inconsistent for some  $k < \omega$ .

$p(x)$   *$p$ -divides over*  $A$  if  $p(x)$  strongly divides over  $Ac$  for some parameter  $c$ .

Weak notions of  $p$ -dividing could be defined in many ways. By the definition,  $p$ -dividing implies dividing. So we expect that weak  $p$ -dividing implies weak dividing.

**Definition 10** Let  $A \subset B$  and  $p(x) \in S(B)$ .

$p(x)$  *weakly  $p$ -divides over*  $A$  if there is a formula  $\varphi(\bar{x}) = \exists y \bigwedge_{i < n} \psi(x_i, y) \in L_n(A)$  such that  $[p \upharpoonright A]^\varphi$  is consistent, while  $[p]^\varphi$  is inconsistent.

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